Measuring Default Correlation

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Preliminaries

- Default Correlation is the greater tendency for any two firms to jointly default together than one would infer from their individual likelihoods of defaulting

- How do we define default?
  - Failure to pay
  - Restructuring
  - Filing for bankruptcy under chapter 11 or chapter 7
  - Repudiation or moratorium (sovereign entities)

- How do we measure default in academic studies?
  - Delisting from exchange codes as reported in CRSP or deletion codes as reported in Compustat

Pricing Credit Risk (1)

- Credit Exposure – how much I am now owed or may be owed from a counterparty or instrument issuer with risk of default.
  - This amount may be stochastic (OTC derivative contract or loan commitment) or fixed (corporate bond or term loan)
- Assume $1 of exposure with a tenor of 1 year with a price of $0.90 today. This price can be viewed as either a <u>risky</u> discount factor or a continuously compounded yield of 10.54%.
- With a risk free price for the same exposure of $0.95, we can further split the risky yield into <u>risk free</u> yield of 5.13% and a <u>risky</u> spread of 5.41%
- This spread compensates us for illiquidity, taxes, default risk and any investor risk aversion

Pricing Credit Risk (2)

- We can also view the same instrument through the lens of a risk neutral investor who only concerns him or herself with expected credit losses
- Thus, the $0.10 discount to par of $1 is the discounted compensation for expected credit loss:

$$e^{-(\text{RiskFree}+\text{Spread})} = e^{-\text{RiskFree}}(1-\text{E}(\text{Credit Loss}))$$

$$e^{-\text{Spread}} = 1-\text{E}(\text{Credit Loss})$$

$$e^{-\text{Spread}} - 1 = -\text{E}(\text{Credit Loss})$$

$$\text{Spread} \approx \text{E}(\text{Credit Loss})$$

Pricing Credit Risk (3)

- Define Credit Loss as an indicator of default accompanied by a loss amount:

$$\text{Credit Loss} = 1_{\{\text{Default?}\}} * \text{Loss}$$

- Further, assume that when default occurs, all of the exposure is lost (the entire $1) and substitute into the final equation on the prior slide:

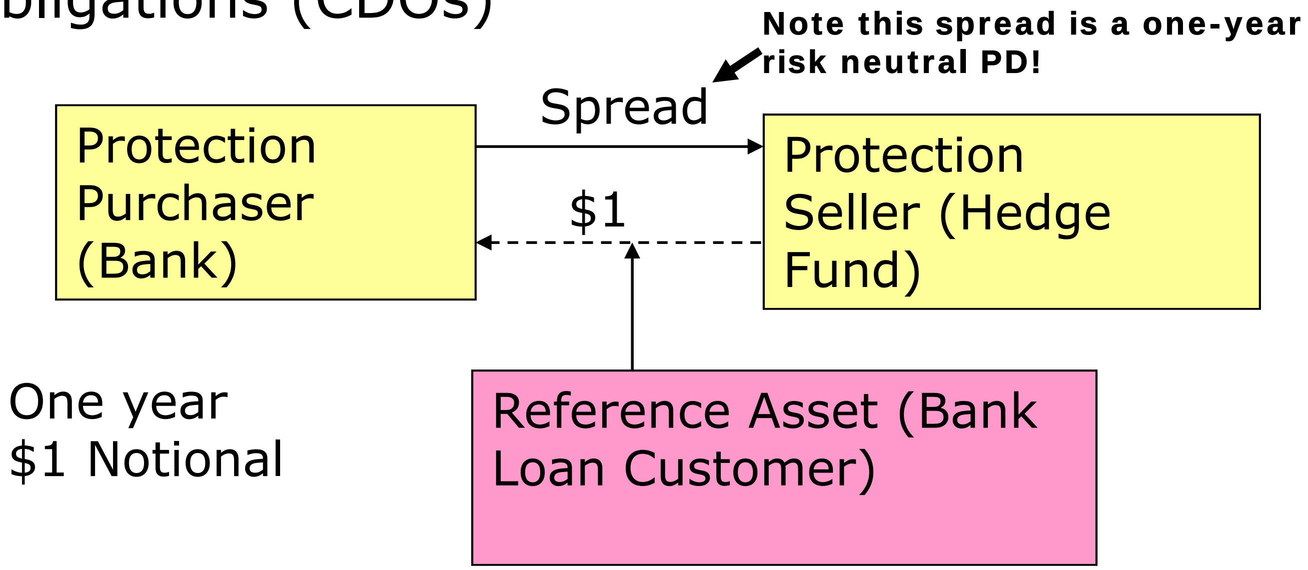
$$\text{Spread} \approx \text{E}\,(\text{Credit Loss})$$

$$\text{Spread} \approx \text{E}\,(1_{\{\text{Default?}\}})$$

- This expectation is the risk neutral likelihood or probability of default (PD). It is larger than the true physical default likelihood (adjusted for the assumption of no recovery in the default event). The default event will be modeled as a Bernoulli random variable over a fixed time period.

Credit Default Swap

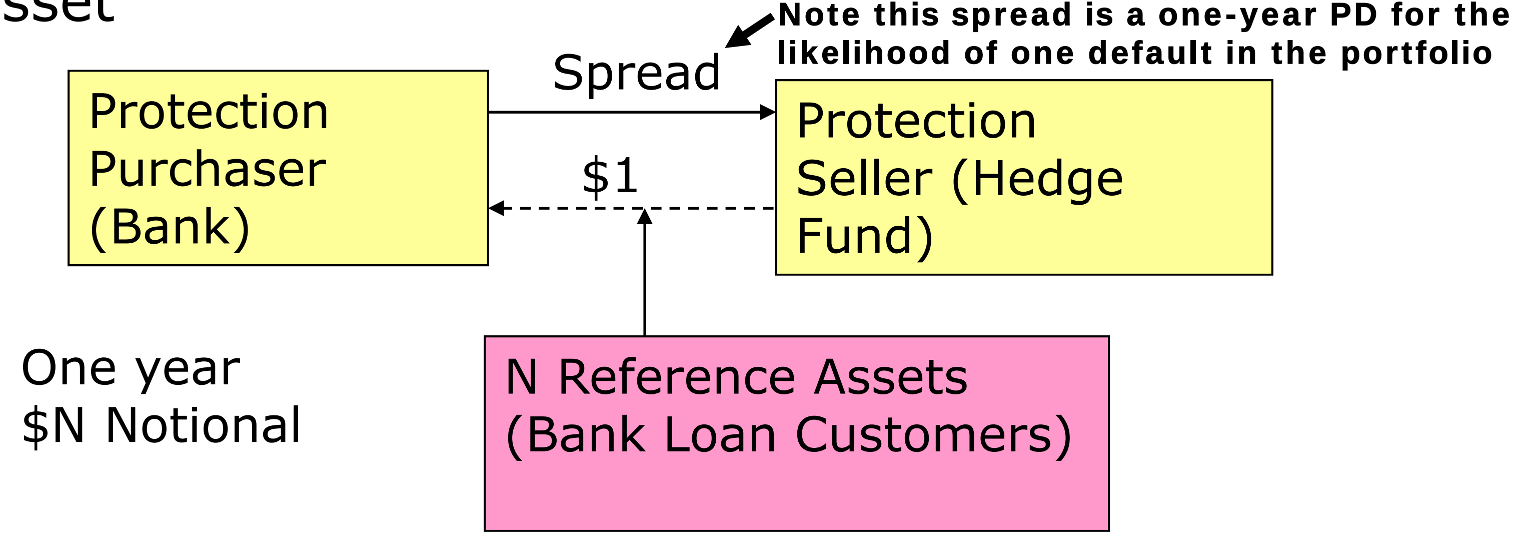
- The building block of multi-name credit derivatives such as first to default baskets and Collateralized Debt Obligations (CDOs)



- Real world issues we will ignore
  - Typical tenors 3 - 5 years paid quarterly
  - Notionals in $Millions
  - Senior unsecured recovery (~$0.40) implies payment LT $1
  - Counterparty Risk matters (introduces default correlation)

First-to-Default Basket

- More than one reference asset each $1 notional so for a basket of N, $N total notional, however in the event that one defaults, reverts to a default swap on the defaulting asset



- We now need to introduce default correlation because the pricing depends upon the relation between firms defaulting

Default Correlation

- Begin with Pearson's correlation coefficient and introduce the Bernoulli default assumption for each of two assets:

$$\rho_{\text{Default}} = \frac{\text{Cov}(x, y)}{\sqrt{\text{Var}(x)\text{Var}(y)}} = \frac{\text{E}(xy) - \text{E}(x)\text{E}(y)}{\sigma_x \sigma_y}$$

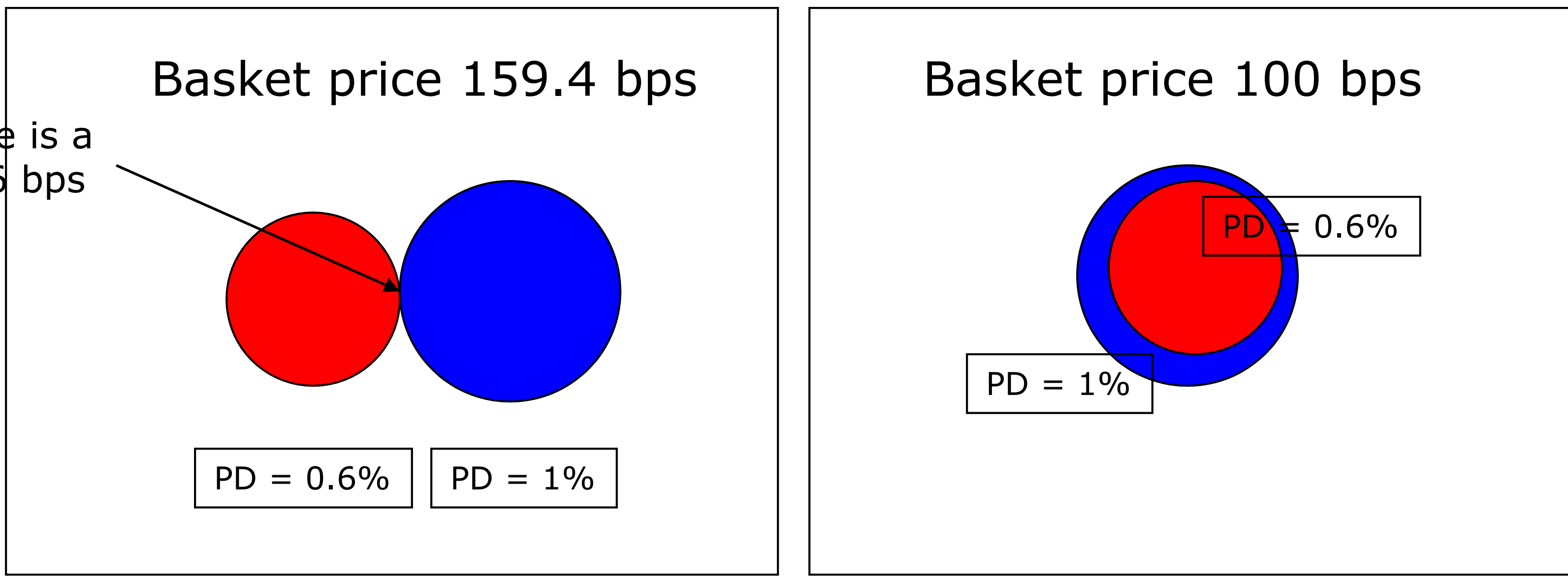
$$= \frac{\text{JPD}_{x,y} - \text{PD}_x \text{PD}_y}{\sqrt{\text{PD}_x(1-\text{PD}_x)\text{PD}_y(1-\text{PD}_y)}}$$

- Note that default correlation is a function of the joint probability of default and the individual probability of default for the two assets

First-to-Default Basket, revisited

- Assume just two assets with annual market default swap spreads of 100 and 60 basis points. Consider two extremes, independence or default correlation of zero and default correlation of one.

Note there is a JPD of 0.6 bps



$\rho_{def} = 0$ $\rho_{def} = 1$

First-to-Default Basket, revisited

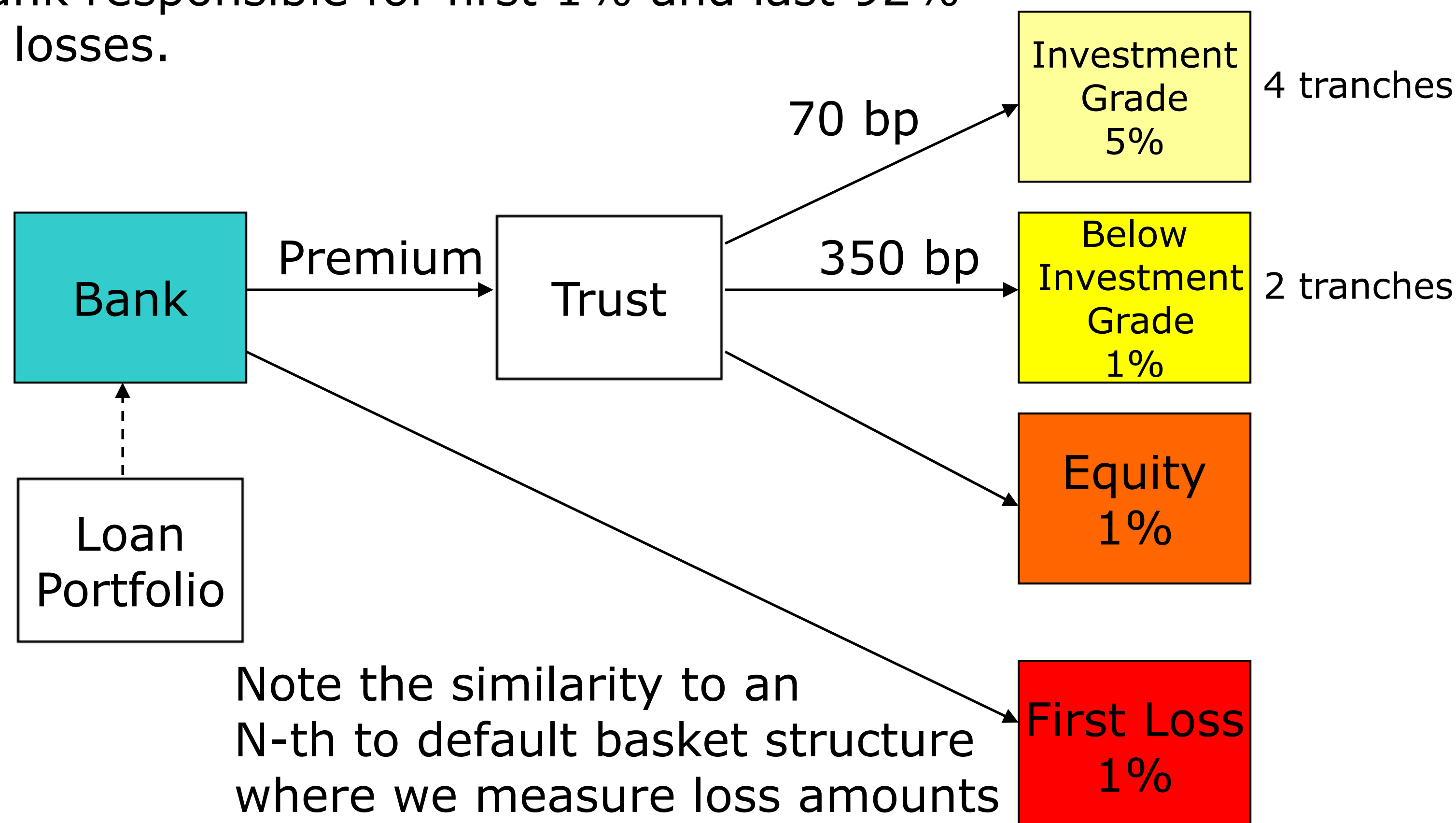
- We can see that the extremes vary from a near additive PD/Spread (100 bp + 60 bp) to the largest of the individual assets (100 bp). <u>These rules hold for larger size baskets as well</u>.

Joint PD	One Default	Implied Cost (bp)	Default Correlation
0.006%	1.594%	**159.4**	0.000%
0.072%	1.528%	**152.8**	8.589%
0.138%	1.462%	**146.2**	17.179%
0.204%	1.396%	**139.6**	25.768%
0.270%	1.330%	**133.0**	34.357%
0.336%	1.264%	**126.4**	42.946%
0.402%	1.198%	**119.8**	51.536%
0.468%	1.132%	**113.2**	60.125%
0.534%	1.066%	**106.6**	68.714%
0.600%	1.000%	**100.0**	77.304%

Note that calculated default correlation is less than one because the two assets differ in individual default likelihood

Generalizing to CDOs (Collateralized Debt Obligations)

Investors pay in 7% of notional amount (funded transaction). Bank responsible for first 1% and last 92% of losses.



Note the similarity to an N-th to default basket structure where we measure loss amounts rather than the number of defaults.

Empirical Measurement

- Rating agency approaches begin with historical averages of single firm default likelihood (PD) based upon credit rating.
- With fixed single firm PDs, cohorts of firms are examined to measure resulting JPDs:

$$\text{JPD} = \frac{\binom{\text{Defaulting Firms}}{2}}{\binom{\text{Total Firms}}{2}}$$

- As default correlation is a function of single firm PD and joint PD, all credit cycle uncertainty is captured in the JPD and thus, in measured default correlation. As single firm PD should also rise and fall with the credit cycle, default correlation is surely overstated.
- Lucas (1995), Bahar and Nagpal (2001), DeServigny and Renault (2003), Akhavein and Kocagil (2005)

Building on Single Firm Approaches: The Structural or Merton Model

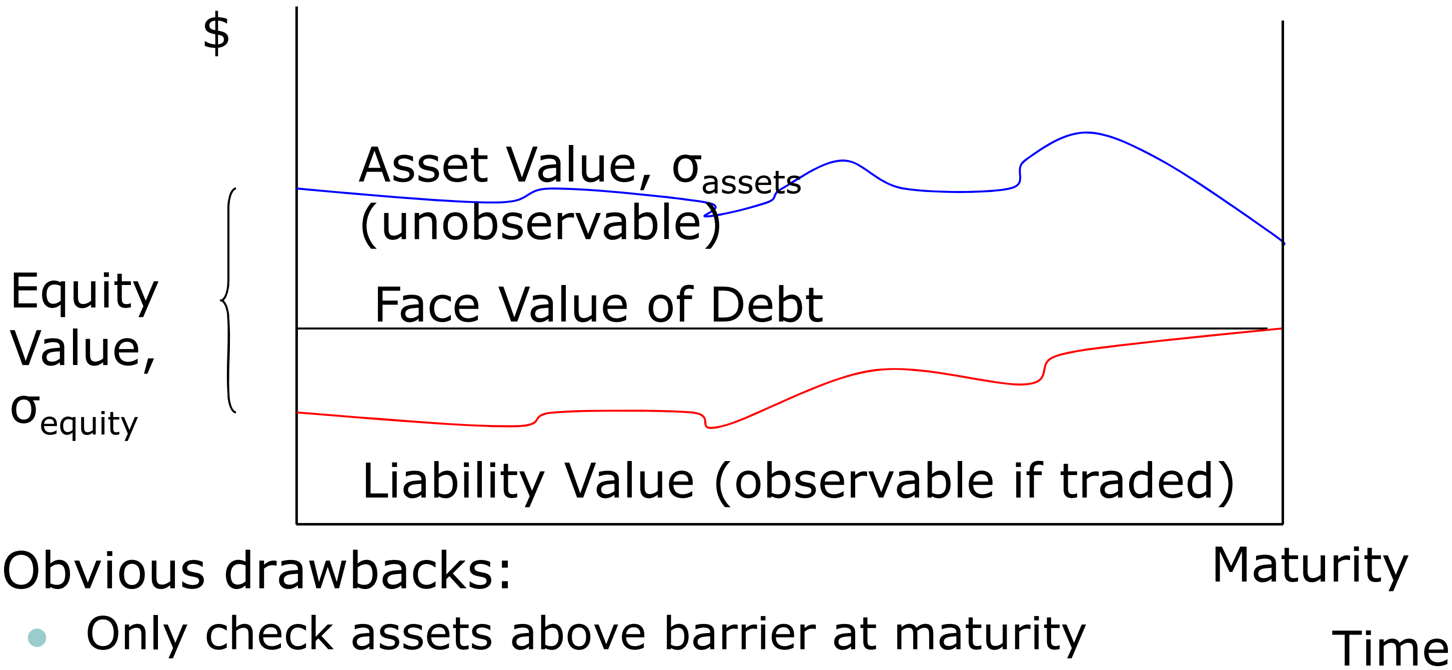
- Merton (1974) proposed an option based approach to risky debt valuation by treating the equity of the firm as a call option on its assets with the strike price being the face value of the debt
- This was a theoretical not an empirical paper
- Major assumptions:
  - Assets follow geometric brownian motion (constant volatility)
  - A single zero coupon debt issue
  - Capital structure is fixed
  - Constant interest rates
- With these assumptions valuation is a straightforward application of BSM for a European style call option where face value of debt is the strike and maturity of the debt determines the maturity of the option. However, from an empirical perspective, while we have an option price, we have one equation in two unknowns – asset value and volatility:

$$\text{Equity} = \text{Assets N}(\text{d}_1) - \text{PV}(\text{X N}(\text{d}_2))$$

$$\text{d}_1 = \frac{\ln(\text{Assets}/\text{X}) + (\text{r} + 0.5\sigma_{\text{Assets}})\text{T}}{\sigma_{\text{Assets}}\sqrt{\text{T}}}$$

$$\text{d}_2 = \text{d}_1 - \sigma_{\text{Assets}}\sqrt{\text{T}}$$

Building on Single Firm Approaches: The Structural or Merton Model



- Obvious drawbacks:
  - Only check assets above barrier at maturity
  - Only one option price observed (the equity price) - multiple liability issues with coupons blur both the implied option maturity and strike price
  - Capital structure will change between now and maturity
  - Off balance sheet activity not captured

Industry Implementation: KMV

- Adds the following result from Ito's Lemma to the BSM equation to create 2 equations in 2 unknowns in solving for asset level and volatility:

$$\sigma_{\text{Equity}} \text{Equity} = \Delta \sigma_{\text{Assets}} \text{Assets}$$

- Major assumptions:
  - Uses historical equity volatility (how measured?)
  - Assumes Face value or Strike price = Short term liabilities + 0.5 * Long term liabilities with a five year term to maturity
  - Capital structure is fixed – debt rolls over
  - Constant interest rates
- With these assumptions calculates Distance to Default

$$\text{DD} = \frac{[\text{Assets - Liabilities}]}{\sigma_{\text{Assets}} * \text{Assets}}$$

- Distance to default is mapped (proprietary) to an empirical database of defaults:
  - 4*sigma is a 1% annual PD vs. zero for Normal (Bohn and Crosbie, 2003)

KMV's Measure of Default Correlation

- As each firm's equity moves, KMV updates its PD for each firm daily. To this is added a measure of asset correlation from a completely independent multifactor model. This second model is static, only updated every 3-4 years
- Joint PD is determined from a bivariate normal distribution with the single firm PDs and the static asset correlation:

$$N_2(N^{-1}(PD_x), N^{-1}(PD_y), \rho_{\text{Asset xy}})$$

- Thus, default correlation is measured:

$$\rho_{\text{Default}} = \frac{N_2(N^{-1}(PD_x), N^{-1}(PD_y), \rho_{\text{Asset xy}}) - PD_x PD_y}{\sqrt{PD_x(1-PD_x)PD_y(1-PD_y)}}$$

- Note how different this is from the rating agency approach. Now all credit cycle influence appears in the single firm measures.
- The joint measure while derived from the market implied assets and empirical distribution through the single firm PD, assumes normality in asset values with a fixed correlation.

The CreditMetrics Approach a.k.a. "KMV-light"

- Developed by JPMorgan in 1997 with input from KMV
- Replaces single firm KMV default model with a constant historical rating agency derived transition matrix
- Replaces KMV asset correlation model with an equity correlation model through loadings on equity indices worldwide
- Keeps the bivariate normal model for assets that KMV introduced

$$N_2(N^{-1}(PD_x), N^{-1}(PD_y), \rho_{\text{Equity x,y}})$$

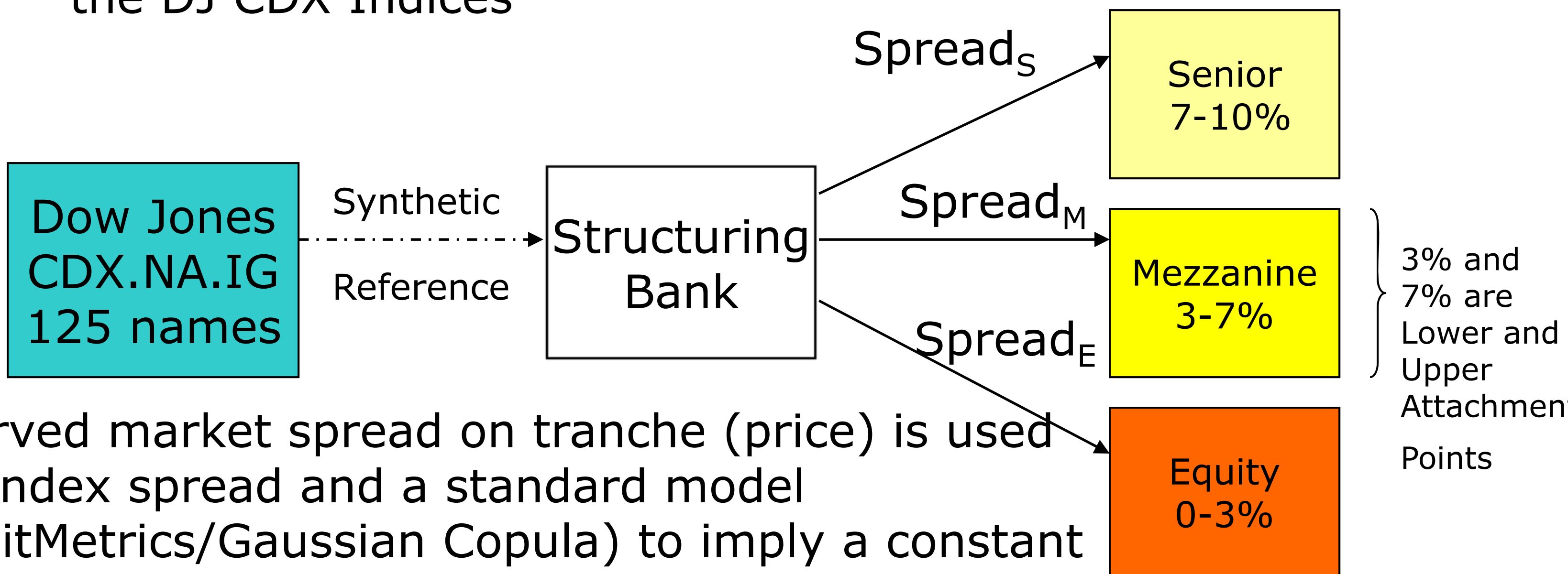
- Thus, default correlation is still measured:

$$\rho_{\text{Default}} = \frac{N_2(N^{-1}(PD_x), N^{-1}(PD_y), \rho_{\text{Equity x,y}}) - PD_x PD_y}{\sqrt{PD_x(1-PD_x)PD_y(1-PD_y)}}$$

- Little dynamism here as single firm PDs do not change, all similarly rated firms (ex. S&P BBB) are identical and equity correlations are typically updated infrequently
- This model is the basis for much of the Basel II Capital Accord developed by Michael Gordy at the Federal Reserve Board

Current Industry Approaches to CDOs build on the CreditMetrics Model

Investors take synthetic loss positions based upon selected "attachment points" referring to a standardized pool such as the DJ CDX Indices



Observed market spread on tranche (price) is used with index spread and a standard model (CreditMetrics/Gaussian Copula) to imply a constant asset correlation analogous to BSM in equity and interest rate markets. Leads to the "compound correlation smile" and "base correlation skew" as successive tranches are fitted to market prices.

Recent Academic Approaches: Hazard Models

- Altman (1968) and Ohlsen (1980) produced the seminal studies in using accounting data to predict single firm bankruptcy.
- Discrete Hazard (Duration) models have several advantages over these early efforts (Shumway, 2001):
  - Automatically adjust for age of the firm (relevant for firms under 10 years of age)
  - Exploit time varying covariates vs. static models
  - Permit macroeconomic variables that will be the same for all firms at a given point in time
  - Utilize more of sample data
- More recent work has differed on the validity of the "KMV type" distance to default measure in the chosen explanatory variables

Recent Academic Approaches: Caveat on KMV distance to default

- KMV's default distance mapping to PD (they use "EDF") is proprietary
- Academic approaches begin with the Merton option model and the iterative solution process pioneered by KMV. However, the distance to default measure is actually a physical drift analogue to d_1 in Merton's model:

$$N\left(-\frac{\ln(\text{Assets/Face})+\left(\mu_{\text{Assets}}-\sigma^2_{\text{Assets}}\right)T}{\sigma_{\text{Assets}}\sqrt{T}}\right)$$

- The standard normal is used for default prediction. While these probabilities will be questionable, the firm to firm ranking should not change under a better mapping.

Hazard Models, cont.

Study	Variables	Out of Sample Accuracy
Hillegeist et al. (2004)	KMV - Dist to Default	N/A
Bharath and Shumway (2004)	Size, ln(Face Debt), Equity Volatility, Excess Return, NI/TA, Naïve Distance to Default, **KMV DD Not meaningful**	80% in first two deciles
Chava and Jarrow (2004)	3 Broad Industries: Excess Return, Relative Size, Equity Volatility,	85.6% in first two deciles
Campbell et al. (2005)	Net Inc to Mkt Val Tot Assets **Avg,** Ex Return **Avg,** Equity Vol, Rel Size, Cash to Mkt Val Tot Assets, Equity Market to Book, Equity Price, **KMV DD not meaningful**	Improves on Bharath /Shumway and Chava/Jarrow
Duffie et al. (2006a)	KMV - Dist to Default, Equity return, 3 month Tbill rate, S&P 500 return	92% in first two deciles

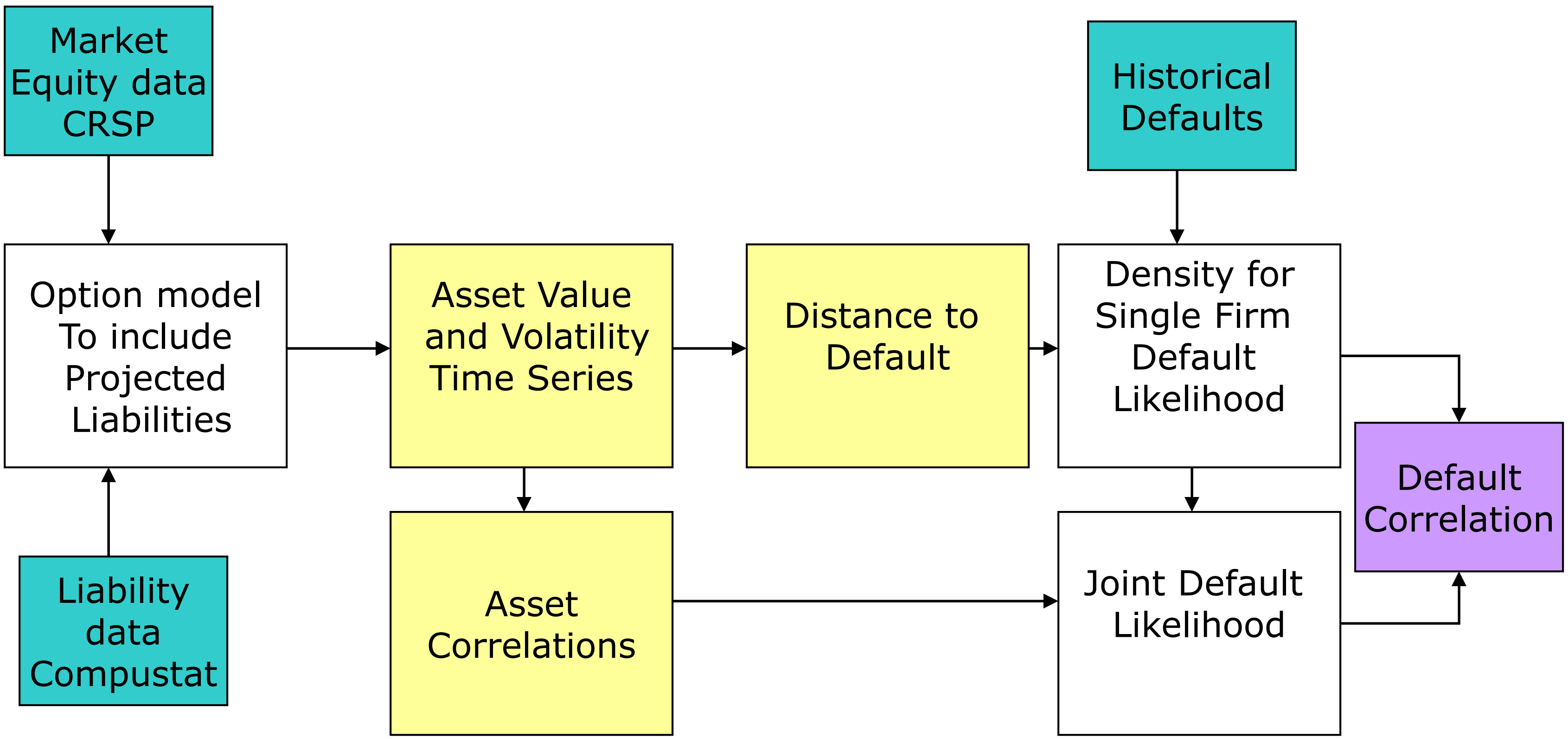
Implications for Default Correlation

- Duffie et al. (2006a) is the most sophisticated of the approaches and has the highest out of sample predictive accuracy
- Duffie et al. (2006b) test and find too much default clustering in the data compared to the predictions of (2006a) to be consistent with the capture of default correlation through common economic factors in the model.
- Duffie et al. (2006c) revisit frailty models to try to capture this missing default correlation
  - Frailty is an unobservable common default risk factor that changes over time
  - They estimate the effect of this unobserved factor is to double the average firm PD from peak to trough of the economic cycle

A Proposed Approach for Measuring Default Correlation

- Build on the Merton model assumptions in existing work
  - Permit capital structure to change
  - Introduce both stochastic asset volatility and a relationship between asset volatility and asset level
- Either repeat the exercises performed in Duffie et al. (2006a and 2006b) or else create an empirical distribution relating modified distance to default to historical default data
  - Asset correlations (cointegration?) may be used to map Joint PD as well as individual firm PD from observed default data

An overview



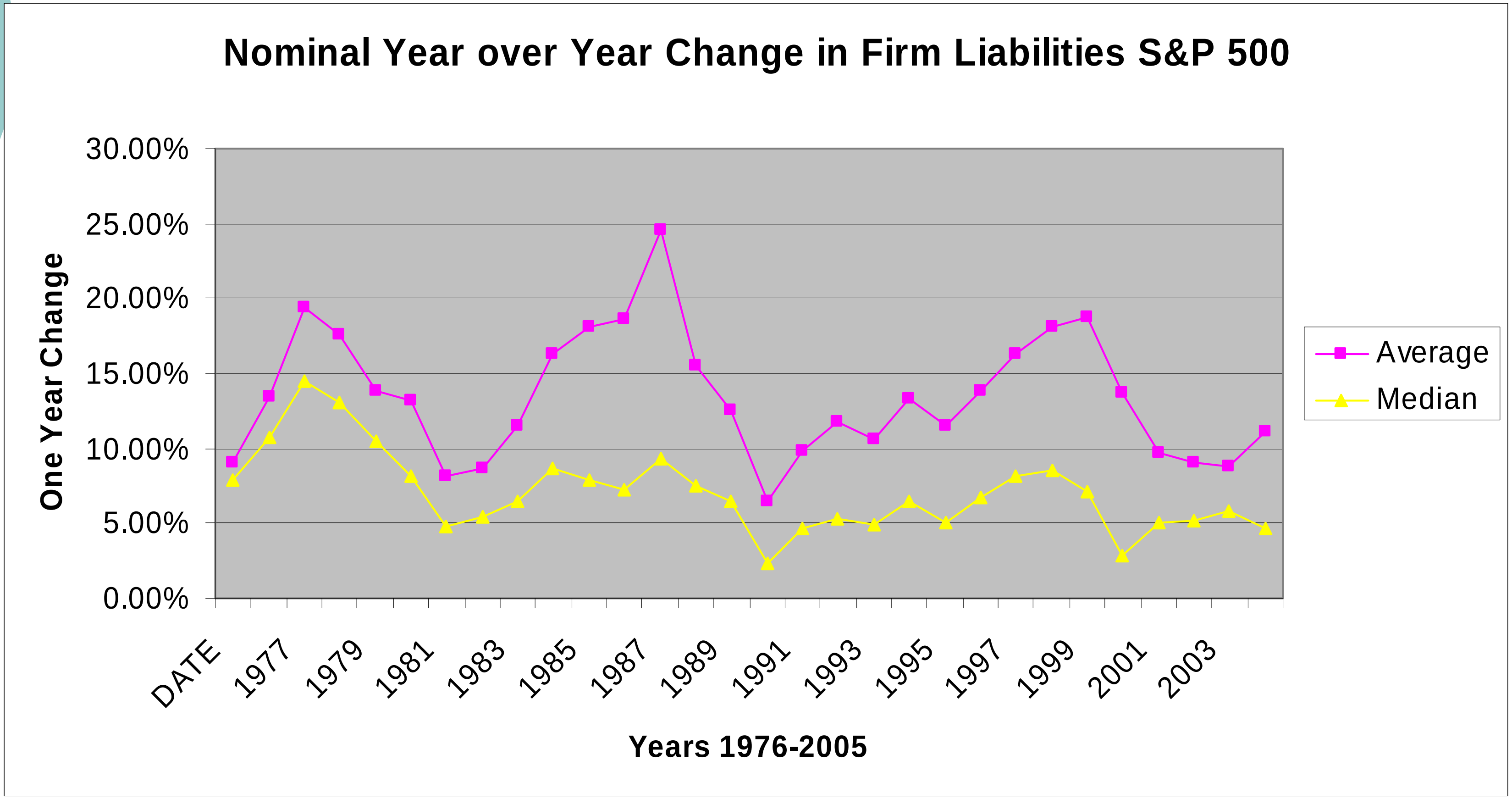
Market Equity data CRSP
Historical Defaults
Option model To include Projected Liabilities
Asset Value and Volatility Time Series
Distance to Default
Density for Single Firm Default Likelihood
Default Correlation
Liability data Compustat
Asset Correlations
Joint Default Likelihood

Changing Capital Structure

- One consistent assumption through all studies is the fixed strike value for debt over the life of the option at the level observed at its start
  - Within the option measure assets drift at the risk free rate and liabilities are fixed
- As asset values are typically modeled with positive drift in the distance to default calculation, the effect is to de-leverage the balance sheet
- Including projected changes in liabilities over both option life and in the distance to default calculation should produce meaningful change in these measures
- What can be said of the growth pattern in liabilities?

Change in Total Liabilities

Compustat item 181 (Current, Other Liab., LT Debt, Def Tax, Min Int)



Other Asset Value Processes

- The existing approach assumes GBM for the assets
- Alternative assumptions are stochastic volatility/GARCH
  - Heston (1993) permits closed-form option prices for European style options under a stochastic volatility model
  - Duan (1995) develops an option pricing model for NGARCH with analytical approximations for European style options
  - Heston and Nandi (2000) develop a closed-form GARCH option pricing model for European style options
  - American style and exotic options can be priced via numerical procedures (Duan and Simonato, 2000 and Ritchken and Trevor, 1999)

GARCH Option Pricing for Unobservable Firm Assets

- Unlike continuous time stochastic volatility models, GARCH is discrete. Thus, the volatility is observable from the asset price history.
- However, in our case, we have no asset price history, as our equity price history is a time series of option prices on the assets of the firm. We wish to back out an asset price history by inverting the option pricing model similar to the Merton approach.
- Further complicating the matter are the 5 parameters for these processes. These include the correlation between asset prices and volatility and the market price of volatility risk.
- Where most studies have a cross section of similar equity options with differing moneyness and maturity, we have a single firm asset option with uncertain moneyness and maturity.

Possible Approaches

- If inferring single firm asset price histories from option price histories proves untenable, group firms into industries and argue that some or all the parameters are industry specific – firms from an industry then provide a cross section of option prices with differing moneyness but similar maturities
- For the subset of data after 1996, can equity options (which in this world are options on options on firm assets) be used to address the market price of volatility risk?
- Other suggestions?

Measuring Individual Default Likelihood

- With the modified time series of firm asset values and volatilities along with an adjusted liability value, we can create new distance to default measures

$$DD = \frac{[\text{Assets - Liabilities}]}{\sigma_{\text{Assets}} * \text{Assets}}$$

- With a database of historical default information, these may be mapped to default likelihoods by estimating a density

Measuring Joint Default Likelihood

- With the time series of asset values, we can measure pair wise correlation
- We know that the joint default likelihood should vary between the product of the individual default likelihoods and the smaller of the individual default likelihoods as correlation moves from 0 to 1
- With our density ($f_?$) this implies two distance to default endpoints

$$\mathrm{DD}_{\rho=0} = f_?^{-1}\left(\mathrm{PD}_{\mathrm{small}} * \mathrm{PD}_{\mathrm{large}}\right) \quad \text{to} \quad \mathrm{DD}_{\rho=1} = f_?^{-1}\left(\mathrm{PD}_{\mathrm{small}}\right)$$

- With the correlations and the empirical default database a mapping from asset correlation to the "joint distance measure" could be performed
- Alternatively, a Multivariate Density could be estimated?